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WEIGHTED L^p CLOSURE THEOREMS FOR SPACES OF ENTIRE FUNCTIONS

BY

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ABSTRACT

For a fixed weight $\Delta(dx)$ on R^1 and a linear space $\mathcal{H} \subseteq L^p(\Delta)$ of entire functions that is closed under difference quotients $h(\cdot) \to (z - \cdot)^{-1}[h(z) - h(\cdot)]$, the $L^p(\Delta)$ closure $\bar{\mathcal{H}}$ of $\mathcal{H}$ is studied and characterized in terms of the norms $L(z)$, $(z \in C^1)$ of the evaluation functionals $h \to h(z)$, $h \in \mathcal{H}$.

1. Introduction

Let $\Delta(dx)$ be a Borel measure on the real line R^1 with finite moments $\int x^n \Delta(dx)$ of all orders. A classical theorem of M. Riesz (see [13] or [2]) shows that if the space $\mathcal{H}$ of polynomials is not dense in $L^2(\Delta)$ then the closure $\bar{\mathcal{H}}$ of $\mathcal{H}$ in $L^2(\Delta)$ contains only entire functions $f(z)$ which satisfy

$$\limsup_{z \to \infty} |z|^{-1} \log |f(z)| = 0. \tag{1.1}$$

This result of Riesz is typical of a class of theorems in the theory of one variable trigonometrical and polynomial approximation each of which asserts that the closure $\bar{\mathcal{H}}$ of a given linear space $\mathcal{H}$ of entire functions either is everything or is a proper subspace of entire functions which satisfy growth conditions similar to (1.1). One can find examples of such theorems in the work of Koosis [6], Krein [9], Levinson and McKean [10] and Pitt [12], and this list is far from complete.

In this paper we investigate this general phenomenon within the setting of weighted $L^p(\Delta)$ approximation. We single out the essential feature common to the spaces $\mathcal{H}$ which occur in the classical examples as being closed under difference quotients: if $f \in \mathcal{H}$ and $\mathrm{Im}\, z \neq 0$ then the function

$$\zeta \to (\zeta - z)^{-1}[f(\zeta) - f(z)] \in \bar{\mathcal{H}}. \tag{1.2}$$

We will however also treat infinite measures for which $L^p(\Delta)$ does not contain the quotient (1.2) unless $f(z) = 0$. To avoid this difficulty we replace (1.2) with

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the weaker condition: If f and g are in $\mathcal{H}$ and $\operatorname{Im} z \neq 0$ then the function

$$\zeta \to (z-\zeta)^{-1}[f(z)g(\zeta)-f(\zeta)g(z)] \in \bar{\mathcal{H}}. \tag{1.3}$$

It develops that condition (1.3) essentially forces $\bar{\mathcal{H}}$ to assume one of three basic forms:

i) $\bar{\mathcal{H}} = L^p(\Delta)$;

ii) $\bar{\mathcal{H}}$ is a space of functions analytic on the upper or lower half-plane and similar in nature to the classical H^p spaces; or

iii) $\bar{\mathcal{H}}$ is a space of entire functions satisfying certain growth conditions.

When the third alternative occurs $\bar{\mathcal{H}}$ exhibits a general structure similar to that of the Hilbert spaces of entire functions which de Branges [3] has studied. This structure theory, although much less complete than de Branges' theory, remains rich enough to be interesting and contains an analogue of the basic inclusion theorem ([3, p. 107]) of de Branges.

Our methods and techniques are intimately related to those of the Bernstein problem [7] and [11], the classical moment problem [2] and especially those of de Branges' Hilbert spaces of entire functions [3]. In particular the proof of our Theorem 5 is entirely derivative from de Branges' work.

A description of the basic results is given in Section 2. The core of the paper is in Sections 3 through 8. Section 9 describes the special structure available when Δ is discrete. Section 10 is a description of spaces of analytic functions on the upper half-plane.

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2. Preliminaries and statement of results

Let $\Delta(dx)$ be a σ-finite Borel measure on R^1 and for $1 \leqq p < \infty$ let $L^p(\Delta)$ be the weighted L^p space of measurable functions $f(x)$ with norm

$$\|f\|_p^p = (\int |f(x)|^p \Delta(dx)) < \infty.$$

$\mathcal{H}$ will denote a fixed linear space of entire functions $f(z)$ whose restrictions to R^1 are contained in $L^p(\Delta)$ and we may consider $\mathcal{H}$ to be a subspace of $L^p(\Delta)$. Associated with $\mathcal{H}$ is the set of common zeros

$$\mathscr{Z} = \{z \in C^1 : h(z) = 0 \quad \text{for each} \quad h \in \mathscr{H}\},$$

and to avoid trivialities we will assume throughout that

$$\Delta\{\mathscr{Z} \cap R^1\} = 0. \tag{2.1}$$

Let $\bar{\mathscr{H}}$ denote the closure in $L^p(\Delta)$ of $\mathscr{H}$. The basic assumption we make on $\mathscr{H}$ is the condition:

(H_w) Whenever f and g are in $\mathscr{H}$ and Im $z \neq 0$ then

$$\zeta \to (z-\zeta)^{-1}[f(z)g(\zeta) - f(\zeta)g(z)] \in \bar{\mathscr{H}}.$$

Sometimes it will be necessary to assume the stronger condition:

(H_s) Whenever f and g are in $\mathscr{H}$ and Im $z \neq 0$ then

$$\zeta \to (z-\zeta)^{-1}[f(z)g(\zeta) - f(\zeta)g(z)] \in \mathscr{H}.$$

The condition H_w has two equivalent useful formulations:

(H_{w_1}) If $z \notin \mathscr{Z}$ is non-real and $f \in \mathscr{H}$ satisfies $f(z) = 0$ then

$$\zeta \to (z-\zeta)^{-1} f(\zeta) \in \bar{\mathscr{H}}.$$

(H_{w_2}) If $z \notin \mathscr{Z}$ is non-real and $f \in \mathscr{H}$ with $f(z) = 0$ then

$$\zeta \to (\alpha - \zeta)\,(z-\zeta)^{-1} f(\zeta) \in \bar{\mathscr{H}} \quad \text{for each} \quad \alpha \in C^1.$$

Replacing $\bar{\mathscr{H}}$ with $\mathscr{H}$ we have two further conditions H_{s_1} and H_{s_2} each of which is equivalent to H_s.

In discussing the closure $\bar{\mathscr{H}}$ of $\mathscr{H}$ in $L^p(\Delta)$ there are two auxiliary functions which play crucial roles. They are

$$L(z) \equiv \sup\{|h(z)| : h \in \mathscr{H} \text{ and } \|h\|_p \leqq 1\}, \tag{2.2}$$

and

$$L^+(z) \equiv \sup\{|h(z)| : h \in \mathscr{H} \text{ and } \|(i+\zeta)^{-1}h(\zeta)\|_p \leqq 1\}. \tag{2.3}$$

These functions will be used analogously to the comparable functions used by Mergelyan in his solution of the Bernstein problem [11]. Clearly $0 \leqq L(z) \leqq L^+(z) \leqq +\infty$ and $L(z) = 0$ iff $z \in \mathscr{Z}$. $L(z)$ is the $L^p(\Delta)$ norm of the evaluation functional $h \to h(z)$ on $\mathscr{H}$ and $L^+(z)$ is the norm of the same functional in $L^p(\Delta^+)$, where Δ^+ is the measure $\Delta^+(dx) = |i+x|^{-p}\Delta(dx)$. Observe that both $L(z)$ and $\log L(z)$ are subharmonic functions on C^1.

We will make use of the fact that all arguments can be reduced to the case that $\mathscr{Z} = \varnothing$. To see this let $\{z_n\}$ be an enumeration of $\mathscr{Z}$. For $z \in \mathscr{Z}$ set $0(z) = k$ if each $h \in \mathscr{H}$ has a zero of order at least k at $\zeta = z$ and some $h \in \mathscr{H}$ has a zero of exact order k at $\zeta = z$. Let $k(z)$ be the Weierstrass product with zeros of order

$0(z_n)$ at $z_n \in \mathscr{Z}$. For each $h \in \mathscr{H}$, $h(z)k^{-1}(z)$ is entire and the map $h(z) \to h'(z) \equiv h(z)\, k^{-1}(z)$ maps $\mathscr{H}$ onto a space $\mathscr{H}'$ of entire functions with

$$\int |h(\zeta)|^p \Delta(d\zeta) = \int |h'(\zeta)|^p \Delta'(d\zeta),\ \Delta'(d\zeta) = |k(\zeta)|^p \Delta(d\zeta).$$

Moreover, $\mathscr{H}'$ satisfies H_w (or H_s) if $\mathscr{H}$ does and $L(z) = |k(z)|\, L'(z)$.

The complex conjugate of z is $\bar{z}$. If $f(z)$ is analytic the conjugate function $\overline{f}(\bar{z})$ is denoted $f^{\times}(z)$. The space $\mathscr{H}$ is called symmetric if $h \in \mathscr{H}$ implies $h^{\times} \in \mathscr{H}$.

The first three theorems show the relationship between the structure of $\bar{\mathscr{H}}$ and the finiteness of the functions $L(z)$ and $L^{+}(z)$.

THEOREM 1. *Suppose $\mathscr{H}$ satisfies* H_w *and that $L^{+}(\beta) = +\infty$ for some $\beta \in R^{2+} = \{z : \operatorname{Im} z > 0\}$. Then for $h \in \mathscr{H}$*

$$\zeta \to (\zeta - z)^{-1} h(\zeta) \in \bar{\mathscr{H}} \text{ for each } z \in R^{2+} \text{ and}$$

$$\zeta \to e^{-it\zeta} h(\zeta) \in \bar{\mathscr{H}} \text{ for each } t \geqq 0.$$

Theorem 1 has the obvious modification for $\beta \in R^{2-}$. Thus if there exist $\beta_+ \in R^{2+}$ and $\beta_- \in R^{2-}$ with $L^{+}(\beta_{\pm}) = +\infty$ we have $e^{it\zeta}\mathscr{H} \subseteq \bar{\mathscr{H}}$ for all t. From this we deduce $\bar{\mathscr{H}} = L^p(\Delta)$. For symmetric $\mathscr{H}$ we have $L^{+}(\beta) = L^{+}(\bar{\beta})$ and

COROLLARY 1. *If $\mathscr{H}$ is symmetric and $L^{+}(\beta) = +\infty$ for some non-real β then $\bar{\mathscr{H}} = L^p(\Delta)$.*

A near converse to Theorem 1 is given by

THEOREM 2. *Suppose $\mathscr{H}$ satisfies* H_w *and that $\bar{\mathscr{H}} \neq L^p(\Delta)$. Then there exists a non-real $\beta \notin \mathscr{Z}$ with $(\beta - \zeta)^{-1}\mathscr{H} \not\subseteq \bar{\mathscr{H}}$. If $\beta \in R^{2+}$ then $L(z)$ is finite and continuous on R^{2+} and* $\log L(z)$ *is locally integrable on $\overline{R^{2+}}$, i.e. $\int_B |\log L(z)|dxdy < \infty$ for each bounded subset $B \subset R^{2+}$.*

COMMENT. If there exist $\beta_+ \in R^{2+}$ and $\beta_- \in R^{2-}$ with $(\beta_+ - \zeta)^{-1}\mathscr{H} \not\subseteq \bar{\mathscr{H}}$ and $(\beta_- - \zeta)^{-1}\mathscr{H} \not\subseteq \bar{\mathscr{H}}$ it follows from Theorem 2 that $L(z)$ is continuous on all of C^1. For symmetric $\mathscr{H}$ we again have a

COROLLARY 2. *If $\mathscr{H}$ is symmetric and $\bar{\mathscr{H}} \neq L^p(\Delta)$ then $L(z)$ is finite and continuous on C^1.*

Theorem 2 is complemented by

THEOREM 3. *If $\mathscr{H}$ satisfies* H_s *and $0 < L(\beta) < +\infty$ for some $\beta \in R^{2+}$ then $L(z)$ is continuous on R^{2+} and* $\log$ *$L(z)$ is locally integrable on $\overline{R^{2+}}$.*

From this we have

COROLLARY 3. *If $\mathscr{H}$ satisfies* H_s *and is symmetric then either $\bar{\mathscr{H}} = L^p(\Delta)$ or*

$L(z)$ is finite and continuous on C^1. If $\bar{\mathcal{H}} = L^p(\Delta)$ and $L(z)$ is finite then the measure $\Delta(dx)$ is discrete.

In Section 9 we give two examples with $\bar{\mathcal{H}} = L^p(\Delta)$ and $L(z)$ finite. For one of these $L^+(z) < \infty$ for all z and for the other $L^+(z) \equiv +\infty$ for Im $z \neq 0$.

Since $L(z)$ is the norm of the evaluation functional $h \to h(z)$, it follows in case $L(z)$ is continuous on R^{2+} that $L^p(\Delta)$ norm convergence of a sequence $\{h_n\}$ in $\mathcal{H}$ implies local uniform convergence of the analytic functions $\{h_n(z)\}$ on R^{2+}. Thus each function $h \in \bar{\mathcal{H}}$ has a unique analytic extension $\tilde{h}(z)$ for $z \in R^{2+}$ with $\tilde{h}(z) = h(z)$ for $h \in \mathcal{H}$, and such that

$$(2.4) \qquad |\tilde{h}(z)| \leqq L(z)\|h\|_p, \qquad z \in R^{2+}.$$

In case $L(z)$ is continuous on C^1 then $\tilde{h}(z)$ is entire and (2.4) holds for all $z \in C^1$. It is this case where $L(z)$ is continuous and $\bar{\mathcal{H}}$ is a closed space of entire functions satisfying H_s which primarily interests us here, but the case with $L(z)$ continuous on R^{2+} and infinite on R^{2-} is briefly discussed in Section 10. The characterization of which entire functions $h(z)$ are in $\bar{\mathcal{H}}$ is given by

THEOREM 4. *Suppose $\mathcal{H} \subseteq L^p(\Delta)$ satisfies H_w and that $L(z)$ is finite and continuous on C^1. Define $\mathcal{K} \supseteq \mathcal{H}$ to be that subspace of $L^p(\Delta)$ of entire functions $f(z)$ for which $f(z)L^{-1}(z)$ is bounded on C^1. For $1 < p < \infty$, $\bar{\mathcal{H}} = \mathcal{K}$ and for $p = 1$, dim $(\mathcal{K}/\bar{\mathcal{H}}) \leqq 1$.*

An example with dim $(\mathcal{K}/\bar{\mathcal{H}}) = 1$ is given in Section 9. Further information about the structure of $\bar{\mathcal{H}}$ is obtained in Section 7. We show that if $\mathcal{Z} = \emptyset$ and $f \in \bar{\mathcal{H}}$ has zeros $\{z_n\} \subset R^{2+}$ the Blaschke product $\Pi(1 - z/z_n)\,(1 - z/\bar{z}_n)^{-1}$ converges and $f_\infty(z) = f(z)B^{-1}(z) \in \bar{\mathcal{H}}$, and if $g \in \mathcal{H}$ then $g(z)f_\infty^{-1}(z)$ is of bounded type† on R^{2+}. Denoting the mean type of $g(z)f_\infty^{-1}(z)$ on R^{2+} with

$$T_+(g) = \limsup_{y \to +\infty} y^{-1} \log |g(iy)f_\infty^{-1}(iy)|$$

we show that sup $\{T_+(g): g \in \mathcal{H}\} < +\infty$.

The basic inclusion theorem of de Branges ([3, p. 107]) has the following analogue. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be closed symmetric subspaces of $L^p(\Delta)$ satisfying H_s. Set $L_i(z) = \sup\{|h(z)|: h \in \mathcal{H}_i;\ \|h\|_p \leqq 1\}$, and suppose both $L_1(z)$ and $L_2(z)$ are continuous.

† An analytic function $h(z)$, $z \in R^{2+}$, is of bounded type on R^{2+} if it is the ratio of two bounded analytic functions on R^{2+}. We will use elementary results about functions of bounded type repeatedly. An excellent introductory source for the basic material is found in chapter 1 of de Branges' book [3].

THEOREM 5. *If $L_1(z)L_2^{-1}(z)$ is finite and zero free on C^1 and if there exist nonzero functions $h_i(z) \in \mathcal{H}_i$ $(i = 1, 2)$ for which $h_1(z)h_2^{-1}(z)$ is of bounded type on R^{2+} then either $\mathcal{H}_1 \subseteq \mathcal{H}_2$ or $\mathcal{H}_2 \subseteq \mathcal{H}_1$.*

COROLLARY 4. *Let $\mathcal{K}$ satisfy H_s. Then the class of all closed symmetric subspaces $\mathcal{H}$ of $\mathcal{K}$ for which $\mathcal{L} = \emptyset$, is totally ordered.*

3. Proof of Theorem 1

We start with the preliminary

LEMMA 3.1. *Let $\mathcal{H} \subseteq L^p(\Delta)$ satisfy H_w and suppose that $L^+(\beta) = +\infty$ for some non-real β. Then*

$$(\beta - \zeta)^{-1}\mathcal{H} \subseteq \bar{\mathcal{H}}. \tag{3.1}$$

PROOF. $\mathcal{H}_\beta \equiv \{h \in \mathcal{H} : h(\beta) = 0\}$ is the kernel of the evaluation functional $h \to h(\beta)$. Because $L^+(\beta)$ is the norm of this functional in $L^p(\Delta^+)$ we see that if $L^+(\beta) = +\infty$ then $\mathcal{H}_\beta$ is dense in the $L^p(\Delta^+)$ closure of $\mathcal{H}$. Thus $L^+(\beta) = +\infty$ implies that for each $f \in \mathcal{H}$,

$$\inf\{\textstyle\int |f(\zeta) - h(\zeta)|^p\, |i + \zeta|^{-p}\Delta(d\zeta) : h \in \mathcal{H}_\beta\} = 0,$$

and since $\mathrm{Im}(\beta) \neq 0$,

$$\inf\left\{\int \left|\frac{f(\zeta)}{\beta - \zeta} - \frac{h(\zeta)}{(\beta - \zeta)}\right|^p \Delta(d\zeta) : h \in \mathcal{H}_\beta\right\} = 0.$$

By H_{w_1}, $(\beta - \zeta)^{-1}\mathcal{H}_\beta \subseteq \bar{\mathcal{H}}$ and thus $(\beta - \zeta)^{-1}f(\zeta) \in \bar{\mathcal{H}}$ or, what is the same, $(\beta - \zeta)^{-1}\mathcal{H} \subseteq \bar{\mathcal{H}}$.

The next lemma is well-known in its operator theory context and makes no use of the special structure of $\mathcal{H}$. Theorem 1 is an immediate consequence of Lemmas 3.1 and 3.2.

LEMMA 3.2. *Let K be a linear subspace of $L^p(\Delta)$. If there is a $\beta \in R^{2+}$ with $(\beta - \zeta)^{-1}K \subseteq \bar{K}$ then for each $z \in R^{2+}$,*

$$(z - \zeta)^{-1}K \subseteq \bar{K}, \tag{3.2}$$

and for each $t \geqq 0$,

$$e^{-it\zeta}K \subseteq \bar{K}. \tag{3.3}$$

PROOF. Let $K^\perp = \{\mathcal{Q}(\zeta) \in L^q(\Delta) : \int k(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta) = 0$ for all $k \in K\}$, $(1/p + 1/q = 1)$ denote the annihilator of K. To prove (3.2) we define for $k \in K$ and $\mathcal{Q} \in K^\perp$ the analytic function

$$F(z)=\int (z-\zeta)^{-1}k(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta), \qquad z\in R^{2+}. \tag{3.4}$$

By the Hahn-Banach theorem it suffices to show $F(z)\equiv 0$ for $z\in R^{2+}$. The mth derivative of $F(z)$ is

$$F^{(m)}(z)=(-1)^m\int (z-\zeta)^{-m-1}k(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta).$$

By assumption $(\beta-\zeta)^{-1}K\subseteq \bar K$ and thus $(\beta-\zeta)^{-n}K\subseteq\bar K$ for $n>0$. Since $\mathcal{Q}\in K^{\perp}$ we see $F^{(m)}(\beta)=0$ for $m\geqq 0$ and hence $F(z)\equiv 0$ because $F(z)$ is analytic.

To prove (3.3) we use the facts that $F(z)=0$ and

$$i(iy-\zeta)^{-1}=\int_0^\infty e^{-t(y+i\zeta)}dt, \qquad y>0$$

to conclude that

$$0=i\int_0^\infty (iy-\zeta)^{-1}k(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta)$$

$$=\int_0^\infty e^{-yt}\left\{\int_{-\infty}^\infty e^{-i\zeta t}k(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta)\right\}dt, \qquad y>0.$$

By the uniqueness theorem for Laplace transforms, $\int_{-\infty}^{\infty}e^{-i\zeta t}k(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta)=0$ a.e. for $t\geqq 0$. But this Fourier transform is continuous since $k(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta)$ is a finite measure and we see

$$\int_{-\infty}^\infty e^{-i\zeta t}k(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta)=0 \text{ for } t\geqq 0.$$

Using the Hahn-Banach theorem again we conclude $e^{-i\zeta t}K\subseteq\bar K$ for $t\geqq 0$ and thus complete the proof.

If $e^{it\zeta}\mathcal{H}\subseteq\bar{\mathcal{H}}$ for all t we then have

$$0\equiv\int e^{it\zeta}h(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta), \text{ for } t\in R^1,\ h\in\mathcal{H} \text{ and } \mathcal{Q}\in\mathcal{H}^{\perp}.$$

By the Fourier uniqueness theorem this implies $h(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta)\equiv 0$. Together with condition (2.1) this gives $\mathcal{Q}(\zeta)=0$ a.e. $[\Delta]$. Thus $\mathcal{H}^{\perp}=\{0\}$ and again by the Hahn-Banach theorem $\bar{\mathcal{H}}=L^p(\Delta)$.

Since Lemma 3.2 has the obvious modification for $\beta\in R^{2-}$ we may state

PROPOSITION 3.3. *If $\mathcal{H}\subseteq L^p(\Delta)$ satisfies* H_w *and if there are points $\beta_+\in R^{2+}$ and $\beta_-\in R^{2-}$ with $L^+(\beta_\pm)=+\infty$* then $\bar{\mathcal{H}}=L^p(\Delta)$.

4. Proof of Theorem 2

We proceed with a series of lemmas. The first is quite well-known but we include it for completeness.

LEMMA 4.1. *Let $\mu(d\zeta)$ be a finite complex signed measure on R^1 and suppose that*

$$F(z) \equiv \int_{-\infty}^{\infty} (z-\zeta)^{-1}\mu(d\zeta)$$

does not vanish identically for $z \in R^{2+}$. Then $F(z)$ is of bounded type on R^{2+} and $\log|F(z)|$ is locally integrable on $\overline{R^{2+}}$.

PROOF. It suffices to show $F(z)$ is of bounded type (i.e. $F(z) = B_1(z)/B_2(z)$ where $B_1(z)$ and $B_2(z)$ are bounded analytic functions on R^{2+}) since for a nonzero bounded function $B(z)$ the fact that $\log|B(z)|$ is locally integrable on $\overline{R^{2+}}$ follows directly from the Poisson-Jensen inequality

$$\log|B(x+i(y+h))| \leqq \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\log|B(t+ih)|}{(x-t)^2+y^2}\,dt, \; y, h > 0.$$

To see that $F(z)$ is of bounded type write $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ where the μ_i are finite non-negative measures and set $F_j(z) = \int (z-\zeta)^{-1}\mu_j(d\zeta)$. Then $F = F_1 - F_2 + i(F_3 - F_4)$ and it suffices to show $F_j(z)$ has bounded type. But $\operatorname{Im} F_j(z) \leqq 0$ for $z \in R^{2+}$. Thus $B_j(z) = [F_j(z) - i]^{-1}$ is bounded on R^{2+} and $F_j = (1 - iB_j)B_j^{-1}$ is of bounded type on R^{2+}.

From Lemma 3.2 and the subsequent remarks we have

LEMMA 4.2. *If $\mathscr{H} \subseteq L^p(\Delta)$ satisfies H_w and if $\bar{\mathscr{H}} \neq L^p(\Delta)$ then there exists a non-real β with*

$$(\beta-\zeta)^{-1}\mathscr{H} \not\subseteq \bar{\mathscr{H}}.$$

LEMMA 4.3. *Suppose $\mathscr{H} \subseteq L^p(\Delta)$ satisfies H_w and for some $\beta \in R^{2+}$,*

$$(\beta-\zeta)^{-1}\mathscr{H} \not\subseteq \bar{\mathscr{H}}. \tag{4.1}$$

Then $L(z)$ is finite and continuous on R^{2+} and $\log L(z)$ is locally integrable on $\overline{R^{2+}}$.

PROOF. Let $f \in \mathscr{H}$ with $\|f\|_p = 1$ be chosen so that $(\beta-\zeta)^{-1}f \notin \bar{\mathscr{H}}$. By the Hahn-Banach theorem there is a $\mathscr{Q}(\zeta)$ contained in the annihilator $\mathscr{H}^\perp$ of $\mathscr{H}$ with $\|\mathscr{Q}\|_q = 1$ $(1/p + 1/q = 1)$ for which

$$0 \neq \int (\beta-\zeta)^{-1} f(\zeta)\mathscr{Q}(\zeta)\Delta(d\zeta). \tag{4.2}$$

Thus the analytic function

$$F(z) = \int (z-\zeta)^{-1} f(\zeta)\mathscr{Q}(\zeta)\Delta(d\zeta) \tag{4.3}$$

is not identically zero on R^{2+} and by Lemma 4.1, $\log|F(z)|$ is locally integrable on $\overline{R^{2+}}$.

By condition H_w we see that

$$0 \equiv \int (z-\zeta)^{-1}[f(z)g(\zeta)-g(z)f(\zeta)]\mathcal{Q}(\zeta)\Delta(d\zeta), \qquad \operatorname{Im} z \neq 0$$

for each $g \in \mathcal{H}$. Setting

$$G(z) = \int (z-\zeta)^{-1} g(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta) \tag{4.4}$$

we have

$$f(z)G(z) = g(z)F(z), \qquad \operatorname{Im} z \neq 0. \tag{4.5}$$

Since $\| \mathcal{Q} \|_q = 1$ we have the estimate

$$|G(z)| \leqq \frac{1}{y} \| g \|_p, \qquad y = \operatorname{Im} z > 0.$$

Together with (4.5) this gives

$$|g(z)| \leqq \frac{\| g \|_p}{y} \left| \frac{f(z)}{F(z)} \right|, \text{ if } z \in R^{2+} \text{ and } F(z) \neq 0.$$

From the definition of $L(z)$ we then have

$$|F(z)| \leqq L(z) \leqq \frac{1}{y} \left| \frac{f(z)}{F(z)} \right|, \text{ if } z \in R^{2+} \text{ and } F(z) \neq 0. \tag{4.6}$$

The three functions $\log|f(z)|$, $\log y$ and $\log|F(z)|$ are each locally integrable on $\overline{R^{2+}}$ and thus (4.6) shows that $\log L(z)$ is locally integrable on $\overline{R^{2+}}$.

This implies that the subharmonic function $\log L(z)$ is locally bounded above on R^{2+}. Thus $L(z)$ is locally bounded on R^{2+} and the family $\{g(z): g \in \mathcal{H}$ and $\| g \|_p \leqq 1\}$ is locally bounded on R^{2+} and hence is locally equicontinuous on R^{2+}. It follows that

$$L(z) = \sup\{|g(z)|: g \in \mathcal{H} \text{ and } \| g \|_p \leqq 1\}$$

is continuous on R^{2+} and the proof is complete.

Lemma 4.3 is easily modified to the case that for some $\beta \in R^{2-}$, $(\beta-\zeta)^{-1}\mathcal{H} \nsubseteq \bar{\mathcal{H}}$. Combining Lemmas 3.2, 4.2 and 4.3 gives the following proposition which contains Theorem 2 as a special case.

Proposition 4.4. *Suppose $\mathcal{H} \subseteq L^p(\Delta)$ satisfies H_w. Then one of the following alternatives holds*:

i) *for each non-real z,* $(z-\zeta)^{-1}\mathcal{H} \subseteq \bar{\mathcal{H}}$, *in which case* $\bar{\mathcal{H}} = L^p(\Delta)$,

ii) $(z-\zeta)^{-1}\mathcal{H} \subseteq \bar{\mathcal{H}}$ *holds for each* $z \in R^{2-}$ *but for no* $z \in R^{2+}$, *in which case* $L(z)$ *is continuous on* R^{2+} *and* $\log L(z)$ *is locally integrable on* $\overline{R^{2+}}$ *and moreover* $e^{it\zeta}\mathcal{H} \subseteq \bar{\mathcal{H}}$ *holds for all* $t \geqq 0$,

iia) *alternative* (ii) *holds with the roles of* R^{2+} *and* R^{2-} *interchanged*,

iii) *for no non-real z is* $(z-\zeta)^{-1}\mathcal{H} \subseteq \bar{\mathcal{H}}$, *in which case* $L(z)$ *is continuous in the entire complex plane.*

PROOF. The only item needing comment is the continuity of $L(z)$ in (iii). But we know that $\log L(z)$ is locally integrable and the continuity of $L(z)$ follows as in the proof of Lemma 4.3.

COMMENT. The occurrence of alternative (ii) above is familiar and essentially understood within the context of prediction theory and is briefly discussed in Section 10. We also mention that it is possible to have $\bar{\mathcal{H}} = L^p(\Delta)$ with $L(z)$ finite and continuous. See Section 9.

5. Proof of Theorem 3

The proof rests on the immediate

LEMMA 5.1. *Suppose* $\mathcal{H} \subseteq L^p(\Delta)$ *satisfies* H_s. *For* $\beta \in C^1$ *set* $\mathcal{H}_\beta = \{h \in \mathcal{H}: h(\beta) = 0\}$. *Then* $\mathcal{H}_\beta$ *also satisfies condition* H_s. *If* α *and* β *are non-real and neither is in* $\mathcal{Z}$ *then* $(\alpha-\zeta)\mathcal{H}_\beta = (\beta-\zeta)\mathcal{H}_\alpha$.

For technical reasons it is easier to discuss $L^+(\beta)$ than $L(\beta)$ and because of this we start the proof of Theorem 3 with

LEMMA 5.2. *If* $\mathcal{H} \subseteq L^p(\Delta)$ *satisfies* H_s *and if* $0 < L^+(\beta) < \infty$ *for some* $\beta \in R^{2+}$ *then* $\log L(z)$ *is locally integrable on* $\overline{R^{2+}}$.

PROOF. If $0 < L^+(\beta) < \infty$ then $\mathcal{H}_\beta$ is not dense in the $L^p(\Delta^+)$ closure of $\mathcal{H}$. Thus for some $f \in \mathcal{H}$,

$$0 < \inf\left\{\int \left|\frac{h(\zeta)}{(\beta-\zeta)} - \frac{f(\zeta)}{(\beta-\zeta)}\right|^p \Delta(d\zeta) : h \in \mathcal{H}_\beta\right\},$$

and $(\beta-\zeta)^{-1}f(\zeta)$ is not in the $L^p(\Delta)$ closure of $(\beta-\zeta)^{-1}\mathcal{H}_\beta$. From here on we imitate the proof of Lemma 4.3. We choose $\mathcal{Q} \in L^q(\Delta)$ $(1/p + 1/q = 1)$ so that $F(z) = \int (z-\zeta)^{-1}f(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta)$ does not vanish identically on R^{2+} but such that

$$0 = \int (\beta-\zeta)^{-1}h(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta), \quad \text{for} \quad h \in \mathcal{H}_\beta. \tag{5.1}$$

By Lemma 5.1, $(\beta-\zeta)^{-1}\mathcal{H}_\beta = (z-\zeta)^{-1}\mathcal{H}_z$ if $z \notin \mathcal{Z}$ and z is non-real. Hence

$$0 \equiv \int (z-\zeta)^{-1}[f(z)g(\zeta)-g(z)f(\zeta)]\mathcal{Q}(\zeta)\Delta(d\zeta), \qquad z \in R^{2+}$$

and the proof may be completed exactly as in the case of Lemma 4.3.

PROOF OF THEOREM 3. Let

$$\mathcal{K} = \{h \in \mathcal{H} : \int |h(\zeta)|^p \, |i-\zeta|^p \Delta(d\zeta) < \infty\}.$$

$\mathcal{K}$ is a space of entire functions satisfying H_s and for $\beta \in R^{2+} - \mathcal{Z}$,

$$(\beta-\zeta)^{-1}\mathcal{H}_\beta \subseteq \mathcal{K}.$$

Setting $\tilde{L}(z) = \sup\{|k(z)| : k \in \mathcal{K}$ and $\int |k(\zeta)|^p |i-\zeta|^p \Delta(d\zeta) \leqq 1\}$ and applying Lemma 5.2 we see that if $0 < L(\beta) < \infty$ for some $\beta \in R^{2+}$ then $\log \tilde{L}(z)$ is locally integrable on R^{2+}. But then

$$\sup\left\{ \left| \frac{h(z)}{\beta - z} \right| : h \in \mathcal{H}_\beta \quad \text{and} \quad \int |h(\zeta)|^p \, |\beta-\zeta|^{-p} \, |i-\zeta|^p \, \Delta(d\zeta) < 1 \right\} \leqq \tilde{L}(z).$$

Letting $c = \inf\{|\beta-\zeta|^{-p}|i-\zeta|^p : \zeta \in R^1\}$ we see

$$\sup\{|h(z)| : h \in \mathcal{H}_\beta \quad \text{and} \quad \|h\|_p \leqq 1\} \leqq c\,|z-\beta|\,\tilde{L}(z).$$

Now choose an $h \in \mathcal{H}$ with $h(\beta) = 1$. For each $f \in \mathcal{H}$ we can write

$$f(z) = f(\beta)h(z) + (f(z) - f(\beta)h(z)).$$

Then $f(z) - f(\beta)h(z) \in \mathcal{H}_\beta$ and

$$\|f - f(\beta)h\|_p \leqq \|f\|_p(1 + L(\beta)\|h\|_p).$$

Hence

$$L(z) \leqq L(\beta)\,|h(z)| + c\,|z-\beta|\,\tilde{L}(z)\{1 + L(\beta)\|h\|_p\}.$$

This shows $\log L(z)$ is locally integrable on $\overline{R^{2+}}$ and completes the proof of Theorem 3.

REMARKS. To see how Corollary 3 follows note that Theorem 3 implies for symmetric $\mathcal{H}$ satisfying H_s that either $L(z) \equiv \infty$ for non-real $z \notin \mathcal{Z}$ or that $L(z)$ is finite and continuous. If $L(z) \equiv \infty$ then $\bar{\mathcal{H}} = L^p(\Delta)$ by Theorem 1. If $L(z)$ is finite we know that $\bar{\mathcal{H}}$ is a space of entire functions and if in addition $\mathcal{H} = L^p(\Delta)$, each $f \in L^p(\Delta)$ must agree a.e. $[\Delta]$ with the restriction to R^1 of an entire function. In particular f may be chosen continuous and Δ must be discrete.

6. Proof of Theorem 4

As explained in Section 2 we may assume without loss of generality that

$\mathcal{Z} = \emptyset$. Recalling also that $\mathcal{K}$ is the set of entire functions $f(z)$ for which $\|f\|_p < \infty$ and $f(z)L^{-1}(z)$ is bounded we note: If $\Delta_1 \geqq 0$ is a measure with $\int (L(x))^p \Delta_1(dx) \leqq 1$ and if $\Delta_2 = \Delta + \Delta_1$ then the measures Δ and Δ_2 both determine equivalent norms on $\mathcal{H}$ and both determine the same space $\mathcal{K}$. Thus we may assume for the proof of Theorem 4 that Δ is not discrete.

The heart of the proof is contained in

PROPOSITION 6.1. *Suppose $\mathcal{H} \subseteq L^p(\Delta)$ satisfies* H_w *and that $L(z)$ is finite and continuous on C^1. Fix $k \in \mathcal{K}$ with $\|k\|_p \leqq 1$ and $\mathcal{Q} \in \mathcal{H}^\perp \subseteq L^q(\Delta)$ with $\|\mathcal{Q}\|_q \leqq 1$. Then for each $h \in \mathcal{H}$ the entire function*

$$a_h(z) \equiv \int (z-\zeta)^{-1}[h(z)k(\zeta) - k(z)h(\zeta)]\mathcal{Q}(\zeta)\Delta(d\zeta) \tag{6.1}$$

is identically zero.

PROOF. We begin by showing that $A(z) \equiv a_h(z)h^{-1}(z)$ is an entire function. For $h,\ h_1 \in \mathcal{H}$ we know $(z-\zeta)^{-1}[h(z)h_1(\zeta) - h(\zeta)h_1(z)] \in \bar{\mathcal{H}}$ whenever $\operatorname{Im} z \neq 0$. Thus

$$h(z)\int \frac{h_1(\zeta)}{z-\zeta}\,\mathcal{Q}(\zeta)\Delta(d\zeta) = h_1(z)\int \frac{h(\zeta)}{z-\zeta}\,\mathcal{Q}(\zeta)\Delta(d\zeta), \qquad \operatorname{Im} z \neq 0,$$

and if $h_1(z) \neq 0$,

$$a_h(z) = h(z)\int \frac{k(\zeta)}{z-\zeta}\,\mathcal{Q}(\zeta)\Delta(d\zeta) - k(z)\frac{h(z)}{h_1(z)}\int \frac{h_1(\zeta)}{z-\zeta}\,\mathcal{Q}(\zeta)\Delta(d\zeta).$$

Hence

$$A(z) = \int \frac{k(\zeta)}{z-\zeta}\,\mathcal{Q}(\zeta)\Delta(d\zeta) - \frac{k(z)}{h_1(z)}\int \frac{h_1(\zeta)}{z-\zeta}\,\mathcal{Q}(\zeta)\Delta(d\zeta). \tag{6.2}$$

That is,

$$A(z) = a_{h_1}(z)h_1^{-1}(z) \quad \text{if} \quad \operatorname{Im} z \neq 0,\ h_1(z) \neq 0,\ h \neq 0. \tag{6.3}$$

Since $a_h(z)h^{-1}(z)$ is analytic at each point z with $h(z) \neq 0$, and since we may assume $\mathcal{Z} \neq 0$, (6.3) shows that $A(z)$ is entire. To see that $A(z) \equiv a_h(z)h^{-1}(z)$ vanishes identically we will show it is bounded. Two estimates are obvious:

$$\left| \int (z-\zeta)^{-1}k(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta) \right| \leqq \frac{1}{|y|}, \qquad z = x + iy, \tag{6.4}$$

$$\sup\left\{ \left| \int (z-\zeta)^{-1}h_1(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta) \right| : \|h_1\|_p \leqq 1 \right\} \leqq \frac{1}{|y|}. \tag{6.5}$$

Now for a fixed non-real z we may choose $h_1 \in \mathcal{H}$ with $\| h_1 \|_p \leqq 1$ and $| k(z) h_1^{-1}(z) | \leqq 2$. From (6.2) we have

$$| A(z) | \leqq \frac{3}{| y |}, \quad z = x + iy. \tag{6.6}$$

If $A(z)$ does not vanish identically then $\log | A(z) |$ is subharmonic and (6.6) shows by the mean value property

$$\log | A(z_0) | \leqq \frac{1}{\pi} \int\int_{|z - z_0| \leqq 1} \log | A(z) | dxdy$$

for subharmonic functions, that $\log | A(z) |$ is bounded above. Thus $| A(z) |$ is bounded and must be constant. By (6.6) this constant is zero and the proof is complete.

Proof of Theorem 4. Assume $p > 1$ and that Δ is not discrete. If an entire function $k(z)$ vanishes on the support of Δ then $k(z) \equiv 0$ and by the Hahn-Banach theorem it suffices to show for $k \in \mathcal{H}$ and $\mathcal{Q} \in \mathcal{H}^\perp$ that

$$0 = \int k(\zeta) \mathcal{Q}(\zeta) \Delta(d\zeta). \tag{6.7}$$

By Proposition 6.1 we have for $h \in \mathcal{H}$ with $\| h \|_p = 1$,

$$0 = \int \frac{k(\zeta)}{z - \zeta} \mathcal{Q}(\zeta) \Delta(d\zeta) - \frac{k(z)}{h(z)} \int \frac{h(\zeta)}{z - \zeta} \mathcal{Q}(\zeta) \Delta(d\zeta),$$

whenever z is non-real and $h(z) \neq 0$. Thus

$$\begin{aligned} \int \frac{z}{z - \zeta} k(\zeta) \mathcal{Q}(\zeta) \Delta(d\zeta) = & \frac{k(z)}{h(z)} \int h(\zeta) \left[\frac{z}{z - \zeta} - 1 \right] \mathcal{Q}(\zeta) \Delta(d\zeta) \\ & + \frac{k(z)}{h(z)} \int h(\zeta) \mathcal{Q}(\zeta) \Delta(d\zeta). \end{aligned} \tag{6.8}$$

The second term on the right vanishes since $h \in \mathcal{H}$ and $\mathcal{Q} \in \mathcal{H}^\perp$. For $z = iy$ and $\| h \|_p \leqq 1$ we have

$$\left| \int \left[\frac{iy}{iy - \zeta} - 1 \right] \mathcal{Q}(\zeta) h(\zeta) \Delta(d\zeta) \right| \leqq \left\| \left[\frac{iy}{iy - \zeta} - 1 \right] \mathcal{Q} \right\|_q,$$

and thus

$$\left| \int \frac{iy}{iy - \zeta} k(\zeta) \mathcal{Q}(\zeta) \Delta(d\zeta) \right| \leqq \frac{| k(iy) |}{L(iy)} \left\| \left[\frac{iy}{iy - \zeta} - 1 \right] \mathcal{Q} \right\|_q. \tag{6.9}$$

But $| k(z) | L^{-1}(z)$ is bounded by some constant c and since $p > 1$ we have $q < \infty$. Applying the dominated convergence theorem to both sides of (6.9) gives

$$\left| \int k(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta) \right| \leqq c \lim_{y\to\infty} \left\| \left[\frac{iy}{iy-\zeta} - 1 \right] \mathcal{Q} \right\|_q$$

$$= 0,$$

and completes the proof for $p > 1$.

For $p = 1$ this argument fails since for general $\mathcal{Q} \in L^\infty$ it is not true that

$$\left\| \left[\frac{iy}{iy-\zeta} - 1 \right] \mathcal{Q} \right\|_\infty \to 0 \quad \text{as} \quad y \to \infty.$$

What we can conclude is that: If $k \in \mathcal{K}$ and $\liminf |k(iy)| L^{-1}(iy) = 0$ as $|y| \to \infty$ then $k \in \bar{\mathcal{H}}$. If $\dim(\mathcal{K} \mid \bar{\mathcal{H}}) \geqq 2$ there must exist two functions k_1 and k_2 in $\mathcal{K}$ for which $\alpha k_1 + \beta k_2 \in \bar{\mathcal{H}}$ iff $\alpha = \beta = 0$. Since $k_j(z)L^{-1}(z)$ bounded for $j = 1$ and 2, there must exist a sequence $y_n \to \infty$ for which both limits $c = \lim k_1(iy_n)L^{-1}(iy_n)$ and $d = \lim k_2(iy_n)L^{-1}(iy_n)$ exist. Then $cd \neq 0$ since neither k_1 nor k_2 is in $\bar{\mathcal{H}}$. But then

$$(c^{-1}k_1(iy_n) - d^{-1}k_2(iy_n))L^{-1}(iy_n) \to 0 \quad \text{as} \quad n \to \infty$$

which implies $c^{-1}k_1 - d^{-1}k_2 \in \bar{\mathcal{H}}$ and thus $\dim(\mathcal{K} \mid \bar{\mathcal{H}}) \leqq 1$.

7. General structure theorems

In this section we complement Theorem 4 by proving a number of results concerning functions $f(z) \in \bar{\mathcal{H}}$. Throughout this section we will assume that $\mathcal{H} \subseteq L^p(\Delta)$ is closed and satisfies H_s, that $L(z)$ is finite and continuous on C^1 and, for simplicity's sake, that $\mathcal{X} = \varnothing$.

Proposition 7.1. *Let $h \in \mathcal{H}$ be nonzero and denote the zeros of h in R^{2+} by $\{z_n\}$, counting multiplicities. Then the Blaschke product*

$$B(z) = \prod_1^\infty (1 - z/z_n)(1 - z/\bar{z}_n)^{-1}$$

converges for each $z \notin \{\bar{z}_n\}$ and $h(z)B^{-1}(z) \in \mathcal{H}$.

Proof. Set $B_n(z) = \Pi_1^n (1 - z/z_j)(1 - z/\bar{z}_j)^{-1}$. Then $|B_n(x)| = 1$ for $x \in R^1$ and $|B_n(z)| \downarrow$ as $n \uparrow$ for $z \in R^{2+}$. By condition H_{s_2}, $h_n(z) = h(z)B_n^{-1}(z)$ is in $\mathcal{H}$. Since $|h(x)| = |h_n(x)|$ for $x \in R^1$, the continuity of $L(z)$ implies $\{h_n(z)\}$ is locally bounded. Thus for $z \in R^{2+}$ and $z \notin \{z_n\}$, $\lim_{n\to\infty} |B_n(z)|$ exists and is nonzero. It is now elementary to check (see e.g. [3, p. 20]) that this implies $\Sigma y_n (x_n^2 + y_n^2)^{-1} < \infty$ where $z_n = x_n + iy_n$. In turn, this shows the Blaschke product converges for $z \notin \{\bar{z}_n\}$. Thus $h_n(\zeta) \to h(\zeta)B^{-1}(\zeta)$ and $|h_n(\zeta)| = |h(\zeta)|$ for all real ζ. Hence $h_n \to hB^{-1}$ in $L^p(\Delta)$ and $hB^{-1} \in \mathcal{H}$.

PROPOSITION 7.2. *Let f and g be in $\mathscr{H}$. If $g(z)f^{-1}(z)$ is analytic in R^{2+} then $g(z)f^{-1}(z)$ is of bounded type in R^{2+}.*

PROOF. Assuming, as we may, that Δ is not discrete then the containment $(\beta-\zeta)^{-1}\mathscr{H}\subseteq\mathscr{H}$, $\operatorname{Im}\beta\neq 0$, can not hold as a statement about subspaces of $L^p(\Delta)$ without also holding as a statement about spaces of analytic functions. Since $\mathscr{H}$ only contains entire functions it is clear that $(\beta-\zeta)^{-1}f(\zeta)\notin\mathscr{H}$ if $f(\zeta)$ is entire and $f(\beta)\neq 0$. Thus we can find a $\mathscr{Q}\in\mathscr{H}^{\perp}$ for which $F(z)=\int(z-\zeta)^{-1}f(\zeta)\mathscr{Q}(\zeta)\Delta(d\zeta)$ does not vanish identically on R^{2+}. Just as in the proof of Lemma 4.3 we have

$$g(z)f^{-1}(z)=G(z)F^{-1}(z),\qquad \operatorname{Im} z>0,$$

where

$$G(z)=\int(z-\zeta)^{-1}g(\zeta)\mathscr{Q}(\zeta)\Delta(d\zeta).$$

Since $G(z)F^{-1}(z)$ is of bounded type on R^{2+} the result follows.

Proposition 7.1 and 7.2 can be used to give a useful modification of Theorem 4.

THEOREM 4'. *Let $h^{+}\in\mathscr{H}$ be zero free in R^{2+} and $h_{-}\in\mathscr{H}$ be zero free in R^{2-}. (Such functions exist by Proposition 7.1.) Define the space $\tilde{\mathscr{H}}$ of all entire functions $k(z)\in L^p(\Delta)$ for which*

$$\begin{aligned}&k(z)h_{+}^{-1}(z)\ \text{is of bounded type on } R^{2+} \text{ and}\\ &k(z)h_{-}^{-1}(z)\ \text{is of bounded type on } R^{2+} \text{ and}\end{aligned}\tag{7.1}$$

$$\limsup_{|y|\to\infty}|k(iy)|L^{-1}(iy)<\infty.\tag{7.2}$$

Then for $p>1$, $\mathscr{H}=\tilde{\mathscr{H}}$ and for $p=1$, $\dim(\tilde{\mathscr{H}}\,|\,\mathscr{H})\leqq 1$.

PROOF. The proof of Theorem 4 works here except that we must show under these conditions that the function

$$a_h(z)=\int(z-\zeta)^{-1}[h(z)k(\zeta)-k(z)h(\zeta)]\mathscr{Q}(\zeta)\Delta(d\zeta)$$

is identically zero for each $k\in\tilde{\mathscr{H}}$, $h\in\mathscr{H}$ and $\mathscr{Q}\in\mathscr{H}^{\perp}$. To see this, we may prove, as in Proposition 6.1, that $a_h(z)h^{-1}(z)$ is entire and that (6.2) holds for each nonzero $h_1\in\mathscr{H}$. Setting $h_1=h_{+}$, (6.2) shows that $a_h(z)h^{-1}(z)$ is of bounded type on R^{2+}. Similarly setting $h_1=h_{-}$ shows $a_h(z)h^{-1}(z)$ is of bounded type on R^{2-}. For $z=iy$ we deduce from (6.2) and (7.2) that

$$|a_h(iy)|\ |h^{-1}(iy)|=O(|y|^{-1}),\qquad |y|\to\infty.\tag{7.3}$$

By a theorem of Krein ([8] or [3, p. 38]) we may conclude that $a_n \cdot h^{-1}$ is of minimal exponential type. The Paley-Wiener theorem and (7.3) then show $a_n(z)h^{-1}(z) \equiv 0$ and the proof is complete.

Let h_+ and h_- be as in Theorem 4. For $g \in \mathcal{H}$ denote the mean type of $g(z)h_+^{-1}(z)$ on R^{2+} by

$$T_+(g) = \limsup_{y \to +\infty} \frac{1}{y} \log |g(iy)h_+^{-1}(iy)|, \tag{7.4}$$

and the mean type of $g(z)h_-^{-1}(z)$ on R^{2-} by

$$T_-(g) = \limsup_{y \to -\infty} \frac{1}{y} \log |g(iy)h_-^{-1}(iy)|. \tag{7.5}$$

Also define

$$T_+ = \sup\{T_+(g) \colon g \in \mathcal{H}\}, \quad \text{and}$$

$$T_- = \sup\{T_-(g) \colon g \in \mathcal{H}\}.$$

PROPOSITION 7.3.

a) *Both T_+ and T_- are finite.*

b) *If $g \in L^p(\Delta)$ is entire then sufficient conditions that $g \in \mathcal{H}$ are that both gh_+^{-1} is of bounded type on R^{2+} and $T_+(g) < T_+$, and gh_-^{-1} is of bounded type on R^{2-} and $T_-(g) < T_-$.*

PROOF OF a). As in the proof of Proposition 7.2 we can choose a $\mathcal{Q} \in \mathcal{H}^\perp$ such that for $g \in \mathcal{H}$

$$g(z)h_+^{-1}(z) = G(z)H_+^{-1}(z) \quad \text{for} \quad z \in R^{2+},$$

where

$$G(z) = \int (z - \zeta)^{-1} g(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta),$$

and

$$H_+(z) = \int (z - \zeta)^{-1} h_+(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta)$$

and $h_+(z)$ does not vanish identically on R^{2+}. The mean type of $G(z)H_+^{-1}(z)$ is given by (see e.g. [3, p. 26])

$$T_+(g) = \lim_{r \to \infty} \frac{2}{\pi} \frac{1}{r} \int_0^\pi \log |G(re^{i\theta})H_+^{-1}(re^{i\theta})| d\theta.$$

The mean types of $G(z)$ and $h_+(z)$ on R^{2+} are given by

$$\tau_+(G) = \lim_{r\to\infty} \frac{2}{\pi}\frac{1}{r}\int_0^\pi \log|G(re^{i\theta})|d\theta$$

$$= \limsup_{y\to+\infty} \frac{1}{y} \log G(iy)$$

$$\leqq 0.$$

$$\tau_+(H_+) = \lim_{r\to\infty} \frac{1}{2\pi}\frac{1}{r}\int_0^\pi \log|H_+(re^{i\theta})|d\theta$$

$$> -\infty.$$

Thus

$$T_+(g) = \tau_+(G) - \tau_+(H_+)$$

$$\leqq -\tau_+(H_+) < \infty.$$

Similarly $T_-(g) < -\tau_-(H_-) < \infty$, thus completing the proof of a).

Proof of b). Since $T_+(g) > T_+$ we can find an $h \in \mathscr{H}$ that is zero free on R^{2+} with $T_+(g) < T_+(h)$. Then

$$\limsup_{y\to\infty} |g(iy)h^{-1}(iy)| = 0 \quad \text{and} \quad \limsup_{y\to\infty} |g(iy)L^{-1}(iy)| = 0.$$

Similarly we can show $\limsup_{y\to-\infty} |g(iy)L^{-1}(iy)| = 0$ and by Theorem 4′, $g \in \mathscr{H}$.

Remark. Assuming $p > 1$ or that $\mathscr{H} = \mathscr{H}$ when $p = 1$ it is easy to show that for each z_0 there is a unique function $h(z) = h(z_0, z) \in \mathscr{H}$ with $\|h\|_p = 1$ and $\|h(z_0)\| = L(z_0)$. If z_0 is real it can be shown that $h(z)$ has only simple real zeros $\{x_n\}$ and that $T_+(h) = T_+$ and $T_-(h) = T_-$. Associated with h we can define a subspace $\mathscr{H}'$ of $\mathscr{H}$ spanned by the functions

$$h_0 = h \quad \text{and} \quad h_n(z) = (z_0 - z)(x_n - z)^{-1}h(z).$$

$\mathscr{H}'$ satisfies H_s and for $p = 2$ it follows from de Branges' work ([3, p. 55]) that $\dim(\mathscr{H} \mid \bar{\mathscr{H}}') \leqq 1$. It would be useful and interesting if this is true for $p \neq 2$ but we have not been able to prove it.

8. Proof of Theorem 5

$\mathscr{H}_1$ and $\mathscr{H}_2$ are closed symmetric subspaces of $L^p(\Delta)$ each satisfying H_s. Both $L_1(z)$ and $L_2(z)$ are finite and continuous and $L_1L_2^{-1}$ is assumed to be continuous and zero free. Without loss of generality we may also suppose that $\mathscr{Z}_1 = \mathscr{Z}_2 = \varnothing$. Further it is assumed that there are nonzero functions $h_i \in \mathscr{H}_i$ with $h_1(z)h_2^{-1}(z)$ of bounded type on R^{2+}. By Propositions 7.1 and 7.2 it is clear that for any pair of functions $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$, if $h_1 h_2^{-1}$ is analytic on R^{2+} then it is of bounded type. Because $\mathcal{H}_1$ and $\mathcal{H}_2$ are symmetric the same statement holds for R^{2-}.

Now fix a function $h_+ \in \mathcal{H}_1$ that is zero free on R^{2+} and set $h_-(z) = h_+^\times(z) \in \mathcal{H}_1$. For $g \in \mathcal{H}_1 \cup \mathcal{H}_2$ the formulas (7.4) and (7.5) for the mean type $T_\pm(g)$ make sense. Because $\mathcal{H}_1$ and $\mathcal{H}_2$ are symmetric,

$$\sup\{T_+(g): g \in \mathcal{H}_i\} = \sup\{T_-(g): g \in \mathcal{H}_i\}, \qquad i = 1,2.$$

We define $T_1 = \sup\{T_+(g): g \in \mathcal{H}_1\}$ and $T_2 = \sup\{T_+(g) = g \in \mathcal{H}_2\}$. From Proposition 7.3 we see that $T_1 < T_2$ implies $\mathcal{H}_1 \subset \mathcal{H}_2$ and that $T_2 < T_1$ implies $\mathcal{H}_2 \subset \mathcal{H}_1$.

We now consider the case $T_1 = T_2$. For each $h \in \mathcal{H}_1$, $k \in \mathcal{H}_2$ and $\mathcal{Q}_1 \in \mathcal{H}_1^\perp$ with $\| k \|_p \leqq 1$ and $\| \mathcal{Q}_1 \|_q \leqq 1$ we define

$$a(z) = \int (z-\zeta)^{-1}[h(z)k(\zeta) - k(z)\, h(\zeta)]\mathcal{Q}_1(\zeta)\Delta(d\zeta).$$

Just as in the proof of Proposition 6.1 we have for each nonzero $h_1 \in \mathcal{H}_1$ that

$$\text{(8.1)} \quad \frac{a(z)}{h(z)} = \int \frac{k(\zeta)}{z-\zeta}\mathcal{Q}_1(\zeta)\Delta(d\zeta) - \frac{k(z)}{h_1(z)}\int \frac{h_1(\zeta)}{z-\zeta}\mathcal{Q}_1(\zeta)\Delta(d\zeta), \ \operatorname{Im} z \neq 0.$$

Then (8.1) shows that $a(z)h^{-1}(z)$ is entire and of bounded type on both R^{2+} and R^{2-}. Since $T_1 = T_2$ it follows from (8.1) that the mean type of $a(z)h^{-1}(z)$ in each of the half-planes R^{2+} and R^{2-} is nonpositive. Again by Krein's theorem it follows that $a(z)h^{-1}(z)$ is of minimal exponential type. Lastly, we see that (8.1) implies

$$\text{(8.2)} \qquad \left|\frac{a(z)}{h(z)}\right| \leqq \frac{1}{|y|}\{1 + L_2(z)L_1^{-1}(z)\}.$$

Similarly for $k \in \mathcal{H}_2$, $h \in \mathcal{H}_1$ and $\mathcal{Q}_2 \in \mathcal{H}_2^\perp$ with $\|h\|_p \leqq 1$ and $\|\mathcal{Q}_2\|_q \leqq 1$ we define

$$b(z) = \int (z-\zeta)^{-1}[k(z)h(\zeta) - h(z)k(\zeta)]\mathcal{Q}_2(\zeta)\Delta(d\zeta).$$

Proceeding as above we find $b(z)k^{-1}(z)$ is an entire function of minimal exponential type and satisfies

$$\text{(8.3)} \qquad \left|\frac{b(z)}{k(z)}\right| \leqq \frac{1}{|y|}\{1 + L_1(z)L_2^{-1}(z)\}.$$

Call $A(z) = a(z)h^{-1}(z)$ and $B(z) = b(z)k^{-1}(z)$. By (8.2) and (8.3) we have

$$\text{(8.4)} \qquad \min\{|A(z)|, |B(z)|\} \leqq 2|y|^{-1}.$$

But by using a method of Carleman [4], de Branges has shown ([3, p. 107]) that if two entire functions $A(z)$ and $B(z)$ of minimal exponential type satisfy (8.4) then one of them is zero.

Thus we may suppose $a(z) \equiv 0$ for each $h \in \mathcal{H}_1$, $k \in \mathcal{H}_2$ and $\mathcal{Q}_1 \in \mathcal{H}_1^{\perp}$. If $\mathcal{H}_2 \not\subseteq \mathcal{H}_1$ there must exist a $k \in \mathcal{H}_2$ with $k \notin \mathcal{H}_1$. Since $a(z) \equiv 0$, we have

$$\int \frac{iy}{iy - \zeta} k(\zeta)\mathcal{Q}_1(\zeta)\Delta(d\zeta) = \frac{k(iy)}{h(iy)} \int \frac{iy}{iy - \zeta} h(\zeta)\mathcal{Q}_1(\zeta)\Delta(d\zeta). \tag{8.5}$$

Because $k \notin \mathcal{H}_1$ we may choose $\mathcal{Q}_1$ with

$$0 \neq \int k(\zeta)\mathcal{Q}_1(\zeta)\Delta(d\zeta) = \lim_{|y| \to \infty} \int \frac{iy}{iy - \zeta} k(\zeta)\mathcal{Q}_1(\zeta)\Delta(d\zeta).$$

But

$$0 = \int h(\zeta)\mathcal{Q}_1(\zeta)\Delta(d\zeta) = \lim_{|y| \to \infty} \int \frac{iy}{iy - \zeta} h(\zeta)\mathcal{Q}_1(\zeta)\Delta(d\zeta)$$

and by (8.5) we see that

$$\lim_{|y| \to \infty} \left| \frac{h(iy)}{k(iy)} \right| = 0.$$

Thus

$$\lim_{|y| \to \infty} h(iy)L_2^{-1}(iy) = 0 \text{ for each } h \in \mathcal{H}_1.$$

By Theorem 4′, $\mathcal{H}_1 \subseteq \mathcal{H}_2$.

9. Discrete measures

When Δ is discrete and $\mathcal{H} = L^p(\Delta)$ it is possible to obtain an explicit representation for $L(z)$. The formulas obtained are intimately related to classical results on the cardinal series in interpolatory function theory [15]. These formulas enable us to give an example which shows that for $p = 1$ the alternative dim $(\mathcal{K}/\bar{\mathcal{H}}) = 1$ in Theorem 4 can occur and to give other examples with $L(z) < \infty$ but $L^+(z) \equiv +\infty$.

Let Δ be supported on the discrete set $\{x_n\}$ and let $m_n = \Delta\{x_n\} > \infty$. Assume $\mathcal{H} = L^p(\Delta)$ and that $L(z) < \infty$ for all z. Then for each h there is a unique function $k_n(z) \in \mathcal{H}$ with $k_n(x_m) = 0$ for $n \neq m$ and $k_n(x_n) = (m_n)^{-\alpha}$, $(\alpha = 1/p)$. The set $\{k_n\}$ form a basis for $\mathcal{H} = L^p(\Delta)$ with $\| k_n \|_p = 1$. For any finite sum

$$f(z) = \Sigma\, a_n k_n(z), \tag{9.1}$$

we have

$$\|f\|_p^p = \Sigma |a_n|^p < \infty. \tag{9.2}$$

Thus if $\{a_n\}$ is any sequence satisfying (9.2) the inequality $|f(z)| \leqq L(z)\|f\|_p$ and the fact that $L(z)$ is continuous show that the series (9.1) is locally uniformly convergent. Hence (9.1) and (9.2) establish a linear isometry between $\mathscr{H}$ and the space little l^p. By the duality of the spaces l^p and l^q $(p^{-1}+q^{-1}=1)$ it follows that the norm $L(z)$ of the linear functional $f \to f(z)$ is given by

$$L(z) = \begin{Bmatrix} (\Sigma_n |k_n(z)|^q)^{1/q}, \text{ if } p>1 \\ \sup_n |k_n(z)|, \text{ if } p=1. \end{Bmatrix} \tag{9.3}$$

Setting

$$h(z) = (z - x_0)k_0(z)$$

we observe $h(x_n)=0$ for each n. From the two facts that $L(z)<\infty$ and $\mathscr{H} = L^p(\Delta)$ it easily follows that each of the points x_n is a simple zero of $h(z)$. Thus $(\zeta - x_n)^{-1}h(\zeta) \in \mathscr{H}$ and we find.

$$k_n(z) = [m_n^\alpha h'(x_n)(z-x_n)]^{-1}h(z), \qquad \alpha = 1/p.$$

By (9.1) we conclude that each $f \in \mathscr{H}$ has the absolutely convergent expansion

$$f(z) = h(z)\sum_n \left\{\frac{f(x_n)}{(m_n)\alpha h'(x_n)(z-x_n)}\right\}$$

and that

$$L(z) = \sup_n \frac{1}{m_n h'(x_n)|z-x_n|}|h(z)| \quad \text{for} \quad p=1, \tag{9.4}$$

and

$$L^q(z) = \sum_n |m_n^\alpha h'(x_n)(z-x_n)|^{-q}|h(z)|^q \quad \text{for} \quad p>1. \tag{9.5}$$

Thus $L(z)$ is finite and continuous iff $L(i)<\infty$ and we have the

CRITERION. *$L(z)<\infty$ for all z iff*

$$\sum_n \frac{m_n^{1-q}}{|h'(x_n)(i-x_n)|^q} < \infty \quad \textit{for} \quad p>1, \tag{9.6}$$

and

$$\sup_n \frac{1}{|m_n h'(x_n)(i-x_n)|} < \infty \quad \textit{for} \quad p=1. \tag{9.7}$$

The construction leading to (9.6) and (9.7) is also easily reversible. Starting with a nonzero entire function $h(z)$ that has a simple zero at each of the real points $\{x_n\}$ we let Δ be a discrete measure concentrated on the set $\{x_n\}$ with $m_n = \Delta\{x_n\} > 0$ for each n. Setting $\mathcal{H}$ equal to the linear manifold of linear combinations of the functions

$$h_n(\zeta) = (\zeta - x_n)^{-1} h(\zeta),$$

we observe $\mathcal{H}$ is dense in $L^p(\Delta)$. The identity

$$(\zeta - z)^{-1}\{h_n(\zeta)h_m(z) - h_m(\zeta)h_n(z)\} = h_n(z)(z - x_m)^{-1}[h_n(\zeta) - h_m(\zeta)]$$

shows that $\mathcal{H}$ satisfies H_s. The equations (9.4) and (9.5) follow as before and the criterion enables one to determine if $L(z) \equiv +\infty$ or if $L(z) < \infty$.

EXAMPLES. Let $h(x) = \pi^{-1}\sin(\pi x)$. Then $x_n = n$ and $h'(n) = (-1)^n$. Fix a real λ and set $m_n = |i - n|^\lambda$. Then the criterion reduces in this case to:

$$L(z) < \infty \text{ for all } p > 1 \text{ iff } \lambda > -1$$

$$L(z) < \infty \text{ for } p = 1 \text{ iff } \lambda \geqq -1.$$

In this case the measure Δ^+ is also discrete with $\Delta^+(n) = m_n^+ = |i - n|^{\lambda - p}$. We see that for $p > 1$ a necessary and sufficient condition that $L(z) < \infty$ but $L^+(z) \equiv \infty$ for $z \notin \{x_n\}$ is that λ satisfy $-1 < \lambda \leqq p - 1$.

This example can also be easily modified to show that the alternative $\dim(\mathcal{H}/\bar{\mathcal{H}}) = 1$ in Theorem 4 can occur when $p = 1$. To see this start with the above example with $m_n = |i - n|^{-1}$ and $p = 1$. Then $L(z) < \infty$ is continuous. We now observe that if $\Delta_1(dx)$ is any finite measure for which $\int L(x)\Delta_1(dx) \leqq 1$ then the two measures $\Delta(dx)$ and $\tilde{\Delta}(dx) = \Delta(dx) + \Delta_1(dx)$ determine equivalent L^1 norms on the space $\mathcal{H}$ for which $\tilde{L}(z) \equiv \sup\{h(z): h \in \mathcal{H} \text{ and } \int |h(x)|\tilde{\Delta}(dx) < 1\}$ satisfies $1/2\ L(z) \leqq \tilde{L}(z) \leqq L(z)$. A glance at (9.4) now shows that $L(z) \geqq \pi^{-1}\ |\sin(\pi z)|$ and hence $\sin(\pi z) \in \tilde{\mathcal{H}}$ where $\tilde{\mathcal{H}} = \mathcal{H}$ consists of all entire functions $h \in L^1(\tilde{\Delta})$ for which $h(z)L^{-1}(z)$ is bounded. If Δ_1 is not concentrated on the integers we see $\dim(\tilde{\mathcal{H}}/\mathcal{H}) \geqq 1$ since $\sin(\pi z) \notin \mathcal{H}$, and by Theorem 4, $\dim(\tilde{\mathcal{H}}/\mathcal{H}) = 1$.

COMMENT. When $p = 2$ and $L(z) < \infty$ de Branges' work ([3, p. 55]) shows the existence of a discrete measure $\nu(dx)$ for which

$$\int |h(x)|^2\Delta(dx) \equiv \int |h(x)|^2\nu(dx) \text{ for each } h \in \mathcal{H}$$

and for which $\bar{\mathcal{H}} = L^2(\nu)$. Thus $\bar{\mathcal{H}}$ and $L(z)$ can always be described within the present context. This possibility does not exist when $p \neq 2$ because the $L^p(\Delta)$ norms $\| h \|_p$ for $h \in \mathcal{H}$ usually determine Δ.

For an example we take the simple case with $\mathcal{H}$ consisting of all linear polynomials and $\Delta(dx)$ a finite measure with compact support. If p is not an integer and $h = 1 + \alpha x$ then for sufficiently small real α

$$\| h \|_p^p = \sum_0^\infty \binom{p}{j} \int x^j \Delta(dx) \alpha^j.$$

This shows Δ is determined by the norms $\| h \|_p$ since these norms determine the moments $\int x^j \Delta(dx)$ and Δ has compact support. When p is an odd integer the above argument fails but the conclusion holds for other reasons. When $p = 2n$ is an even integer the norms $\| h \|_p$ do not, in general, determine Δ.

10. Spaces of analytic functions on R^{2+}

In Proposition 4.4 we observed the possibility that $(\beta - \zeta)^{-1}\mathcal{H} \subseteq \bar{\mathcal{H}}$ can hold for each $\beta \in R^{2-}$ but for no $\beta \in R^{2+}$. This situation corresponds to a special case of the invariant subspace theory as described e.g. in Helson's book [5] or Srinivasan and Wang's article [14]. Here we will briefly discuss this correspondence starting from a slightly more general setting than that used in our earlier sections.

The following assumptions will be made throughout this section. $\mathcal{H}$ is a space of analytic functions $h(z)$ defined on R^{2+} that have boundary values $h(x)$ in the sense that

(10.1a) $$\lim_{y \downarrow 0} h(x + iy) = h(x) \text{ exist a.e. } [\Delta],$$

and

(10.1b) $$h(x) \in L^p(\Delta) \text{ for each } h \in \mathcal{H}.$$

The condition $\Delta(\mathcal{Z} \cap R^1) = 0$ will be replaced with

(10.2) $$\text{If } \int_B |h(x)|^p \Delta(dx) = 0 \text{ for each } h \in \mathcal{H}, \text{ then } \Delta(B) = 0.$$

We also assume that condition H_w holds for each $z \in R^{2+}$ and that

(10.3) $$(\beta - \zeta)^{-1}\mathcal{H} \subseteq \bar{\mathcal{H}} \text{ for each } \beta \in R^{2-}.$$

Defining $L(z)$ for $z \in R^{2+}$ as before we can easily modify the proofs in Sections 3 and 4 to give

LEMMA 10.1. *For each* $t \geqq 0$, $e^{it\zeta}\mathcal{H} \subseteq \bar{\mathcal{H}}$. *If* $\bar{\mathcal{H}} \neq L^p(\Delta)$ *then* $L(z)$ *is continuous on* R^{2+} *and* $\log L(z)$ *is locally integrable on* $\overline{R^{2+}}$.

Let $\Delta(dx) = \Delta_s(dx) + \Delta_a(x)dx$ be the Lebesgue decomposition of Δ into a singular part Δ_s and an absolutely continuous part $\Delta_a(x)dx$. The role of the singular part may be understood in terms of the next three lemmas.

LEMMA 10.2. *If* $\bar{\mathcal{H}} \neq L^p(\Delta)$ *then* $\Delta_a(\zeta) > 0$ *a.e.* $[d\zeta]$.

PROOF. If $\bar{\mathcal{H}} \neq L^p(\Delta)$ it follows from (10.2) that there exists an $h \in \mathcal{H}$ and a $\mathcal{Q} \in \mathcal{H}^\perp$ so that the measure $\mu(d\zeta) = h(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta)$ is not zero. But for $t \geqq 0$ we have $e^{it\zeta}h \in \bar{\mathcal{H}}$ and thus $0 = \int e^{it\zeta}\mu(d\zeta)$ for $t \geqq 0$. By the F. and M. Riesz theorem μ has the form $\mu(d\zeta) = f(\zeta)d\zeta$ where $|f(\zeta)| > 0$ a.e. $[d\zeta]$. Thus $\Delta_a(\zeta) > 0$ a.e. $[d\zeta]$.

LEMMA 10.3. *If* $\bar{\mathcal{H}} \neq L^p(\Delta)$ *and* $\mathcal{H}_\infty = \cap\{e^{it\zeta}\bar{\mathcal{H}} : t \geqq 0\}$ *then*

$$\mathcal{H}_\infty = \{f \in L^p(\Delta) : \| f \|_p^p = \int |f(\zeta)|^p \Delta_s(d\zeta)\}.$$

Thus $\mathcal{H}_\infty$ *is naturally identified with* $L^p(\Delta_s)$.

PROOF. For each real t, $e^{it\zeta}\mathcal{H}_\infty \subseteq \mathcal{H}_\infty \subseteq \bar{\mathcal{H}}$. Thus if $\mathcal{Q} \in \mathcal{H}^\perp$ is nonzero and $k \in \mathcal{H}_\infty$ we see $0 \equiv \int e^{it\zeta}k(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta)$. By Fourier uniqueness, $k(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta) \equiv 0$. But by the proof of Lemma 10.2, $|\mathcal{Q}(\zeta)|\Delta(d\zeta) \approx d\zeta$. Hence $k(\zeta) = 0$ a.e. $[d\zeta]$ and $k \in L^p(\Delta_s)$ which shows $\mathcal{H}_\infty \subseteq L^p(\Delta_s)$.

On the other hand if $k \in L^p(\Delta_s)$ and $\mathcal{Q} \in \mathcal{H}^\perp$ then $0 = \int k(x)\mathcal{Q}(x)\Delta(dx)$. Thus $\{L^p(\Delta_s) + \bar{\mathcal{H}}\}^\perp = \mathcal{H}^\perp$. By the Hahn-Banach theorem $L^p(\Delta_s) \subseteq \bar{\mathcal{H}}$ and $L^p(\Delta_s) = \cap\{e^{it\zeta}\bar{\mathcal{H}} : t > 0\} = \mathcal{H}_\infty$.

If $\bar{\mathcal{H}} \neq L^p(\Delta)$ then $L(z)$ is continuous on R^{2+} and we know that corresponding to each $h(x) \in \bar{\mathcal{H}}$ there is a unique analytic function $\tilde{h}(z)$ defined on R^{2+} satisfying $h(z) = \tilde{h}(z)$ for $h \in \mathcal{H}$ and

$$|\tilde{h}(z)| \leqq \| h \|_p L(z). \tag{10.4}$$

LEMMA 10.4. *If* $k \in \mathcal{H}_\infty = L^p(\Delta_s)$ *then* $\tilde{k}(z) \equiv 0$ *for* $z \in R^{2+}$.

PROOF. Choose $h \in \mathcal{H}$ and $\mathcal{Q} \in \mathcal{H}^\perp$ so that $\int (z-\zeta)^{-1}h(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta)$ is not identically zero on R^{2+}. By condition H_w,

$$h(z)\int (z-\zeta)^{-1}k(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta) = k(z)\int (z-\zeta)^{-1}h(\zeta)\mathcal{Q}(\zeta)\Delta(d\zeta). \tag{10.5}$$

But $k(\zeta) = 0$ a.e. $[d\zeta]$ and $\mathcal{Q}(\zeta)\Delta(d\zeta)$ is absolutely continuous. The left side of (10.5) thus vanishes identically and $\tilde{k}(z) \equiv 0$ for $z \in R^{2+}$.

An interesting consequence of Lemma 10.4 is that if $\bar{\mathscr{H}} \neq L^p(\Delta)$ then

$$L(z) = \sup\{|h(z)| : h \in \mathscr{H} \text{ and } \int |h(\zeta)|^p \Delta_a(\zeta) d\zeta \leqq 1\}. \tag{10.6}$$

We now describe the absolutely continuous case when $\Delta_s \equiv 0$. Assuming $\bar{\mathscr{H}} \neq L^p(\Delta)$ then starting with a nonzero $h \in \mathscr{H}$ we may proceed exactly as in Section 7 and form a convergent Blaschke product $B(z)$ containing the zeros of h. Setting $h_\infty(z) = h(z)B^{-1}(z)$ we can show that $h_\infty \in \bar{\mathscr{H}}$ and that for each $k \in \bar{\mathscr{H}}$, $\tilde{k}(z)h_\infty^{-1}(z) = \tilde{k}(z)B(z)h^{-1}(z)$ is of bounded type on R^{2+}. The map $k(z) \rightarrow h_\infty^{-1}(z)k(z)$ is an isometry of $\mathscr{H}$ onto the space $\mathscr{H}_1 \equiv h_\infty^{-1}(z)\mathscr{H} \subseteq L^p(\Delta_1)$, where $\Delta_1(x) = h_\infty^p(x)\Delta(x)$. $\mathscr{H}_1$ satisfies all the assumptions made on $\mathscr{H}$ and in particular, $e^{it\zeta}\mathscr{H}_1 \subseteq \bar{\mathscr{H}}_1$ and $\bar{\mathscr{H}}_1 \neq L^p(\Delta_1)$. By a theorem of Akhiezer [1] it follows that $\log \Delta_1(x)(1+x^2)^{-1}$ is Lebesgue summable. It follows that $\Delta_1(x)$ can be written in the form $\Delta_1(x) = |O(x)|^p$ where $O(z)$ is an outer function in the Hardy space H^p of analytic functions on R^{2+}. The map $k(z) \rightarrow O(z)h_\infty^{-1}(z)k(z)$ is an isometry of $\mathscr{H} \subseteq L^p(\Delta)$ into a subspace $\mathscr{H}_2 = O(z)h_\infty^{-1}(z)\mathscr{H}$ of $L^p(dx)$. Moreover $\bar{\mathscr{H}}_2 \neq L^p(dx)$ and $e^{it\zeta}\mathscr{H}_2 \subseteq \bar{\mathscr{H}}_2$ for $t \geqq 0$. It follows easily from the work of Srinivasan and Wang [14] that $\bar{\mathscr{H}}_2 = O(x)h_\infty^{-1}(x)\bar{\mathscr{H}}$ has the form $\bar{\mathscr{H}}_2 = j_1(x)H^p$, where $|j_1(x)| = 1$ a.e. $[dx]$. Finally since $\bar{\mathscr{H}}_2$ only contains analytic functions of bounded type on R^{2+} we see $j_1(x)$ must have the form $j_1 = j_2 j_3^{-1}$ where j_2 and j_3 are both inner functions and j_3 is zero free on R^{2+}. Thus $\bar{\mathscr{H}} = O^{-1}(z)h_\infty(z)j_2(z)j_3(z)^{-1}H^p$.

In the general picture the sole significance of the factor $k(z) = O^{-1}(z)h_\infty(z)j_2(z)j_3(z)^{-1}$ is that it is an analytic function on R^{2+} that has nonzero boundary values $k(x)$ a.e. $[dx]$ and $|k(x)|^{-p} = \Delta(x)$. Thus the general closed subspace $\mathscr{H} \subseteq L^p(\Delta)$ which satisfies our assumptions is described by the analytic function $k(z)$ with $|k(x)|^{-p} = \Delta(x)$ and $\mathscr{H} = kH^p$.

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